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SIMPLIFIED METHOD OF EVALUATING INTEGRALS OF POWERS OF COSINE USING REDUCTION FORMULA AS MATHEMATICAL ALGORITHM

Lito E. Suello
Malayan Colleges Laguna, Pulo-Diezmo Road, Cabuyao City, Laguna, Philippines
Corresponding Author: lesuello@mcl.edu.ph

Abstract

A simpler and shorter method of evaluating the integrals of powers of cosine is presented in this paper. Generalized formulas in evaluating integrals of odd and even powers of cosine were derived by repeatedly applying the reduction formula for cosine to the integral of the nth power of cosine. From the behavior of the coefficients and exponents of the terms of the derived formulas, algorithms were developed. The new method was compared with the traditional method and results showed that the new method was simpler and shorter. The derived formulas and developed algorithms will be very useful in the study of higher mathematics courses and in many engineering applications.

Keywords: Integration, Mathematical Algorithm, Powers of Cosine, Reduction Formula, Trigonometric Identities

1. Introduction

One of the essential topics in the study of Integral Calculus is evaluating integrals of powers of trigonometric functions. The traditional methods usually use trigonometric identities to transform powers of trigonometric functions into a form where direct integration formulas can be applied. The identity used depends on whether the power is odd or even. For odd powers of cosine, the identity $\cos^2 x = 1 - \sin^2 x$ is applied. The integrand is transformed by factoring out one cosine and the remaining even powered cosine is converted into sine using the identity. The integral is then evaluated using a power formula with the factored cosine used as the differential of sine. For even powers of cosine, the double angle identity $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ is used to reduce the power of cosine into an expression where appropriate integration formulas can already be applied (Dampil, 2014; Dampil, 2015; Hass, Weir, & Thomas, 2014).

Another method used to evaluate powers of cosine is by using a reduction formula. A reduction formula transforms the integral into an integral of the same or similar expression with a lower integer exponent (Riley, Hobson, & Bence, 2010). It is repeatedly applied until the power of the last term is reduced to two or one and the final integral can be evaluated. Using integration by parts, the reduction formula for cosine is (Stewart, 2014).

$$\int \cos^n ax\,dx = \frac{1}{a(n)}\cos^{n-1} ax \sin ax + \frac{n-1}{n}\int \cos^{n-2} ax\,dx$$

The methods discussed above are normally tedious and time consuming depending on the given power of cosine. As shown in the studies of Dampil (Stewart, 2012; Suello , 2015), deriving generalized formulas can simplify solutions. The study of Varberg, Purcell and Rigdon (2014) also revealed that the reduction formula for sine can be generalized and a simpler algorithm can be developed to evaluate integrals of powers of sine. The objective of this paper is to extend the same concept to the integrals of powers of cosine. Generalized formulas were derived by the repeated application of the reduction formula to the integral of the nth power of cosine. The behavior of the coefficients and exponents of the terms in the derived formulas were used as the basis for developing a simpler algorithm.

2. Derivation of Formulas

Given: $\int \cos^n ax dx$, where n is any integer

Using the reduction formula,

$$\int \cos^n ax dx = \frac{1}{a(n)} \cos^{n-1} ax \sin ax + \frac{n-1}{n} \int \cos^{n-2} ax dx$$

Applying the reduction formula to the last term

$$\int \cos^n ax dx = \frac{1}{a(n)} \cos^{n-1} ax \sin ax + \frac{n-1}{n} \left[\frac{1}{a(n-2)} \cos^{n-3} ax \sin ax + \frac{n-3}{n-2} \int \cos^{n-4} ax dx \right]$$

Applying the reduction formula again,

$$\int \cos^n ax dx = \frac{1}{a(n)} \cos^{n-1} ax \sin ax + \frac{n-1}{a(n)(n-2)} \sin^{n-3} ax \cos ax +$$

$$\frac{(n-1)(n-3)}{n(n-2)} \left[\frac{1}{a(n-4)} \cos^{n-5} ax \sin ax + \frac{n-3}{n-4} \int \cos^{n-6} ax dx \right]$$

Simplifying,

$$\int \cos^n ax dx = \frac{1}{a(n)} \cos^{n-1} ax \sin ax + \frac{n-1}{a(n)(n-2)} \cos^{n-3} ax \sin ax +$$

$$\frac{(n-1)(n-3)}{a(n)(n-2)(n-4)} \cos^{n-5} ax \sin ax + \frac{(n-1)(n-3)}{n(n-2)(n-4)} \int \cos^{n-6} ax dx$$

The same trend continues until the last term becomes

$\int \cos ax dx$ if n is odd, or

$\int \cos^2 ax dx$ if n is even

2.1 Odd Powers

$$\int \cos^n ax dx = \frac{1}{a(n)} \cos^{n-1} ax \sin ax + \frac{n-1}{a(n)(n-2)} \cos^{n-3} ax \sin ax +$$

$$\frac{(n-1)(n-3)}{a(n)(n-2)(n-4)} \cos^{n-5} ax \sin ax + ... + \frac{(n-1)(n-3)(n-5)...(2)}{n(n-2)(n-4)(n-6)...(3)} \int \cos ax dx$$

Integrating the last term,

$$\int \cos^n ax dx = \frac{1}{a(n)} \cos^{n-1} ax \sin ax + \frac{n-1}{a(n)(n-2)} \cos^{n-3} ax \sin ax + \frac{n-1}{a(n)(n-2)} \cos^{n-3} ax \sin ax +$$

$$\frac{(n-1)(n-3)}{a(n)(n-2)(n-4)} \cos^{n-5} ax \sin ax + ... + \frac{(n-1)(n-3)(n-5)....(2)}{a(n)(n-2)(n-4)(n-6)...(3)} \sin ax + C$$

Factoring out the common factor gives the formula,

$$\int \cos^n ax dx = \frac{\sin ax}{a} \left[\frac{1}{n} \cos^{n-1} ax + \frac{n-1}{n(n-2)} \cos^{n-3} ax + \frac{(n-1)(n-3)}{n(n-2)(n-4)} \cos^{n-5} ax + ... + \right.$$

$$\left. \frac{(n-1)(n-3)(n-5)...(2)}{n(n-2)(n-4)(n-6)...(3)} \right] + C$$

It can also be written as

$$\int \cos^n ax dx = \frac{\sin ax}{a} \left[C_0 \cos^{n-1} ax + \sum_{j=1}^{\frac{n-1}{2}} C_j \cos^{n-2j-1} ax \right] + C$$

where: $C_0 = \frac{1}{n}$ and $C_j = C_{j-1} \left[\frac{n-2j+1}{n-2j} \right]$

2.2 Even Powers

$$\int \cos^n ax dx = \frac{1}{a(n)} \cos^{n-1} ax \sin ax + \frac{n-1}{a(n)(n-2)} \cos^{n-3} ax \sin ax +$$

$$\frac{(n-1)(n-3)}{a(n)(n-2)(n-4)} \cos^{n-5} ax \sin ax + ... + \frac{(n-1)(n-3)(n-5)...(3)}{n(n-2)(n-4)(n-6)...(4)} \int \cos^2 ax dx$$

Applying the reduction formula to the last term,

$$\int \cos^n ax dx = \frac{1}{a(n)}\cos^{n-1} ax \sin ax + \frac{n-1}{a(n)(n-2)}\cos^{n-3} ax \sin ax + \frac{(n-1)(n-3)}{a(n)(n-2)(n-4)}\cos^{n-5} ax \sin ax$$

$$+ \ldots + \frac{(n-1)(n-3)(n-5)\ldots(3)}{n(n-2)(n-4)(n-6)\ldots(4)}\left[\frac{1}{a(2)}\cos ax \sin ax + \frac{1}{2}\int \cos^0 ax dx\right]$$

Simplifying,

$$\int \cos^n ax dx = \frac{1}{a(n)}\cos^{n-1} ax \sin ax + \frac{n-1}{a(n)(n-2)}\cos^{n-3} ax \sin ax + \frac{(n-1)(n-3)}{a(n)(n-2)(n-4)}\cos^{n-5} ax \sin ax$$

$$+ \ldots + \frac{(n-1)(n-3)\ldots(3)}{a(n)(n-2)(n-4)\ldots(2)}\cos ax \sin ax + \frac{(n-1)(n-3)\ldots(3)}{a(n)(n-2)(n-4)\ldots(2)}x + C$$

Factoring out the common factor gives,

$$\int \cos^n ax dx = \frac{\sin ax}{a}\left[\frac{1}{n}\cos^{n-1} ax + \frac{n-1}{n(n-2)}\cos^{n-3} ax + \frac{(n-1)(n-3)}{n(n-2)(n-4)}\cos^{n-5} ax + \ldots + \right.$$

$$\left.\frac{(n-1)(n-3)(n-5)\ldots(3)}{n(n-2)(n-4)(n-6)\ldots(2)}\cos ax\right] + \frac{(n-1)(n-3)(n-5)\ldots(3)}{n(n-2)(n-4)(n-6)\ldots(2)}x + C$$

The formula may also be written as,

$$\int \cos^n ax dx = \frac{\sin ax}{a}\left[C_0 \cos^{n-1} ax + \sum_{j=1}^{\frac{n-2}{2}} C_j \cos^{n-2j-1} ax\right] + C_{\frac{n-2}{2}} x + C$$

where: $C_0 = \frac{1}{n}$ and $C_j = C_{j-1}\left[\frac{n-2j+1}{n-2j}\right]$

3. Development of the Algorithm for the New Method

A simpler and easier procedure can be developed from the observed trends of the coefficients and exponents of the derived formulas. These are summarized as follows:

3.1 Odd Powers

- Write $\frac{\sin ax}{a}$. This will be followed by a series of cosine terms. For example, $\int \cos^5 2x dx$

$$\frac{\sin 2x}{2}$$

- The first term of the series has a coefficient of $\frac{1}{n}$ and the exponent of cosine is n-1. This coefficient and exponent will be used in determining the coefficient and exponent of the next term.

$$\frac{\sin 2x}{2}\left[\frac{1}{5}\cos^4 2x + \right.$$

- For the next term, the coefficient has a numerator equal to the product of the exponent and the numerator of the preceding term. The denominator is the product of the denominator and exponent minus one of the preceding term. The exponent of cosine is the exponent of the preceding term minus two.

$$\frac{\sin 2x}{2}\left[\frac{1}{5}\cos^4 2x + \frac{(1)(4)}{(5)(3)}\cos^2 2x\right.$$

- Follow the same procedure until the exponent of cosine becomes zero which terminates the series.

$$\frac{\cos 2x}{2}\left[\frac{1}{5}\cos^4 2x + \frac{4}{15}\cos^2 2x + \frac{(4)(2)}{(15)(1)}\cos^0 2x\right]$$

- Add a constant of integration.

$$\int \cos^5 2x dx = \frac{\sin 2x}{2}\left[\frac{1}{5}\cos^4 2x + \frac{4}{15}\cos^2 2x + \frac{5}{15}\right] + C$$

3.2 Even Powers

- Write $\frac{\sin ax}{a}$. This will be followed by a series of cosine terms. For example, $\int \cos^6 3x dx$

$$\frac{\sin 3x}{3}$$

- The first term of the series has a coefficient of $\frac{1}{n}$ and the exponent of cosine is n-1. This coefficient and exponent will be used in determining the coefficient and exponent of the next term.

$$\frac{\sin 3x}{3}\left[\frac{1}{6}\cos^5 3x + \right.$$

- For the next term, the coefficient has a numerator equal to the product of the exponent and the numerator of the preceding term. The denominator is the product of the denominator and exponent minus one of the preceding term. The exponent of sine is the exponent of the preceding term minus two.

$$\frac{\sin 3x}{3}\left[\frac{1}{6}\cos^5 3x+\frac{(1)(5)}{(6)(4)}\cos^3 3x+\right.$$

- Follow the same procedure until the exponent of cosine becomes one which terminates the series.

$$\frac{\sin 3x}{3}\left[\frac{1}{6}\cos^5 3x+\frac{5}{24}\cos^3 3x+\frac{(5)(3)}{(24)(2)}\cos^1 3x\right]$$

- The next term is the product of x and the coefficient of the last term in the cosine series.

$$\frac{\sin 3x}{3}\left[\frac{1}{6}\cos^5 3x+\frac{5}{24}\cos^3 3x+\frac{15}{48}\cos 3x\right]+\frac{15}{48}x$$

- Add a constant of integration.

$$\int\cos^6 3x dx=\frac{\sin 3x}{3}\left[\frac{1}{6}\cos^5 3x+\frac{5}{24}\cos^3 3x+\frac{15}{48}\cos 3x\right]+\frac{15}{48}x+C$$

4. Comparison between the Old and the New Method

Evaluate $\int\cos^7 4x dx$

1. *Using the Old Method*

$$\int\cos^7 5x dx=\frac{1}{5(7)}\cos^6 5x\sin 5x+\frac{6}{7}\int\cos^5 5x dx$$

$$=\frac{1}{35}\cos^6 5x\sin 5x+\frac{6}{7}\left[\frac{1}{5(5)}\cos^4 5x\sin 5x+\frac{4}{5}\int\cos^3 5x\right]$$

$$=\frac{1}{35}\cos^6 5x\sin 5x+\frac{6}{175}\cos^4 5x\sin 5x+\frac{24}{35}\left[\frac{1}{5(3)}\cos^2 5x\sin 5x+\frac{2}{3}\int\cos 5x dx\right]$$

$$=\frac{1}{35}\cos^6 5x\sin 5x+\frac{6}{175}\cos^4 5x\sin 5x+\frac{24}{525}\cos^2 5x\sin 5x+\frac{48}{525}\sin 5x+C$$

$$=\frac{\sin 5x}{5}\left[\frac{1}{7}\cos^6 5x+\frac{6}{35}\cos^4 5x+\frac{8}{35}\sin^2 5x+\frac{16}{35}\right]+C$$

Using the New Method

$$\int\cos^7 5x dx=\frac{\sin 5x}{5}\left[\frac{1}{7}\cos^6 5x+\frac{(1)(6)}{(7)(5)}\cos^4 5x+\frac{(6)(4)}{(35)(3)}\cos^2 5x+\frac{(24)(2)}{(105)(1)}\cos^0 5x\right]+C$$

$$\int \cos^7 5x dx = \frac{\sin 5x}{5}\left[\frac{1}{7}\cos^6 5x + \frac{6}{35}\cos^4 5x + \frac{8}{35}\sin^2 5x + \frac{16}{35}\right] + C$$

Evaluate $\int \cos^4 2x$

Using the Old Method

$$\int \cos^4 2x dx = \frac{1}{2(4)}\cos^3 2x \sin 2x + \frac{3}{4}\int \cos^2 2x dx$$

$$\int \cos^4 2x dx = \frac{1}{8}\cos^3 2x \sin 2x + \frac{3}{4}\left[\frac{1}{2(2)}\cos 2x \sin 2x + \frac{1}{2}\int dx\right]$$

$$\int \cos^4 2x dx = \frac{1}{8}\cos^3 2x \sin 2x + \frac{3}{16}\cos 2x \sin 2x + \frac{3}{8}x + C$$

$$\int \cos^4 2x dx = \frac{\sin 2x}{2}\left[\frac{1}{4}\cos^3 2x + \frac{3}{8}\cos 2x\right] + \frac{3}{8}x + C$$

Using the New Method

$$\int \cos^4 2x dx = \frac{\sin 2x}{2}\left[\frac{1}{4}\cos^3 2x + \frac{(1)(3)}{(4)(2)}\cos 2x\right] + \frac{3}{8}x + C$$

$$\int \cos^4 2x dx = \frac{\sin 2x}{2}\left[\frac{1}{4}\cos^3 2x + \frac{3}{8}\cos 2x\right] + \frac{3}{8}x + C$$

5. Conclusion

The new algorithms developed revealed that the integrals of powers of cosine can be evaluated easily since the tedious repetitions of applying the reduction formula, or expansions of identities using the traditional methods, are eliminated. Integrals can be evaluated directly since the procedure simply involves coefficients and exponents. The derived formulas and algorithms will be very useful in higher mathematics courses like Differential Equations and Advanced Engineering Mathematics and even in the fields of Physics and Mechanics. It can also be used in many engineering applications specifically in electricity and magnetism, waves, heat and mass transfer and reaction kinetics. It is also recommended that the procedure also be applied to the integrals of powers of other trigonometric functions.

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Lito E. Suello is a member of the Mathematical Society of the Philippines and the Philippine Institute of Chemical Engineers. Born in Camarines Sur, Philippines on February 12, 1966, he is a licensed chemical engineer who finished his chemical engineering degree from Central Philippine University in Iloilo City, Philippines in 1987. He also completed master in business administration from San Pedro College of Business Administration, Laguna, Philippines in 2006. At present, he is connected with Malayan Colleges Laguna, Philippines where he teaches mathematics, mechanics, and chemical engineering courses.