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A STOCHASTIC GROWTH PRICE MODEL USING A BIRTH AND DEATH DIFFUSION GROWTH RATE PROCESS WITH EXTERNAL JUMP PROCESS*

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Abstract

One of the most important functions of economists is to provide information on the future of prices, which is important to plan for human activities. There is a considerable interest in stochastic analogs of classical differences and differential equations describing phenomena in theoretical models involving economic structure. In this paper, a description of growth price model using a solution of stochastic differential equation is considered. More specifically, the growth price process follows a Geometric progression with growth rate follow a birth and death diffusion process with random external jump process is studied. The mean and the variance approximation, as well as the predicted and the simulated sample path of such a growth price process are also obtained. Numerical examples for the case of no jumps as well as the case of the occurrence of jump process that follow a uniform and exponential distributions are considered.

Keywords : Growth Price Model, Birth-Death Diffusion Process with Jumps, Growth Rate, Stochastic Differential Equation.

1. Introduction

This paper shows how decision makers' concerns about model specification can affect prices and quantities in a dynamic economy. We use this new approach of stochastic growth process in price models for two reasons. The most important one is this model which is from the type of continuous – time models which is different from what we have already known from the discrete-type models. The second reason is that this kind of model is not widely used in various economic models.

David (1997) studies a model in which production is linear in the capital stocks with technology stocks that have hidden growth rates. Veronesi (1999) studies a permanent income model with a riskless linear technology. Dividends are modeled as an additional consumption endow cent. Hidden information was introduced into asset pricing models by Detemple (1986), who considers a production economy with Gaussian unobserved variables.

Hand (2001) hasdeveloped methods using statistical tools such as logisting regression and naïve Bayes, as well as neural networks for assessing performance of the models to the consumer credit risk.

Al-Eideh and Hasan (2002) have considered growth price models under random environment using a solution of stochastic differential equation of the logistic price model, as well as the logistic price models with random external jump process. They derived the steady state probability and the time dependent probability functions. Also, the mean and the variance as well as the sample path of such a process are considered.

During this past decade, there has been increasing effort to describe various facts of dynamic economic interactions with the help of stochastic differential processes. Thus, stochastic differential processes provide a mechanism to incorporate the influences associated with randomness, uncertainties, and risk factors operating with respect to various economic units (stock prices, labor force, technology variables, etc.)

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Therefore, the techniques of stochastic processes become relevant in pursuing quantitative studies in economics. Stochastic modeling techniques not only enable us to obtain reliable estimates of certain useful economical parameters, but also provides indispensable tools for estimating parameters which are often associated with a high degree of non-sampling errors. Thus, it is proposed to introduce a study of economic structure using the techniques of stochastic processes.

Stochastic differential equation processes have been introduced in the study of three principal categories of economic phenomena: (a) description of growth of certain factors under uncertainty (b) the nature of option price variations presenting certain market conditions and (c) stochastic dynamic programming and control objectives.

Numerous researchers have worked on studying various economic units from different points of view. For example, Aase and Guttrop (1987) studied the role of security prices allocative in capital market, they present stochastic models for the relative security prices and show how to estimate these random processes based on historical price data. The models they suggest may have continuous components, as well as discrete jumps at random time points. Also, two classical applications are Metron (1971) and Black and Scholes (1973). New references include Harrison and Pliska (1981) and Aase (1984). Whereas the first two works only study processes with continuous sample paths, the other two allow for jumps in the paths as well. In other words, the processes have sample paths that are continuous from the right and have left hand limits (in fact, these processes are semi-martingales; for general theory of semi-martingales, see e.g. Kabanov et al., 1979 sec. 2).

Many other authors have studied this problem from different points of view, such as Stein and Stein (1991), Tauchen and Pitts (1983), Schwert (1990), Duffie and Singleton (1993), McGrattan (1996), Callen and Chang (1999), Karmeshu and Goswami (2001), etc.

In this paper, we present new growth price models using a solution of stochastic differential equation. More specifically, the growth price process follows a Geometric progression with growth rate follow a birth and death diffusion process with random external jump process is studied. The mean and the variance approximation, as well as the sample path of such a process are also obtained. The mean and the variance, as well as the predicted and the simulated sample path of such a growth price process are also obtained. Numerical examples for the case of no jumps, as well as the case of the occurrence of jump process that follow a uniform and exponential distributions are considered.

2. The Growth Price Model Using a Birth and Death Diffusion Growth Rate Process With External Jump Process

Now as a matter of fact, modeling's of Natural Prices have three basic characterizations as follows:

(1) The Prices over time may show the average density of price being maintained at a constant level over a long period of time, unless there is a major environmental change.

(2) The growth of a Price need not necessarily remain at a constant level; but the same may fluctuate around a constant mean value randomly.

(3) The third type is again a generalization of the second type, and is given by superim-position of random cycle of oscillation on the type of random variation.

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With this set up, we propose to develop a simple deterministic model. Here, we ignore such factors as environmental conditions etc. while developing a deterministic model. Further, as in the deterministic model, the probabilistic consideration relating to the variation of the price $S(t)$ is ignored, one can reasonably assume $S(t)$ to be continuous variable. In other words, we structure the deterministic model as the following geometric progression:

$$\frac{dS(t)}{dt} = S(t)[B(t) - D(t)] \tag{1}$$

where $B(t)$ and $D(t)$ are the instantaneous increasing (birth) and decreasing (Death) rates at time t. Note that they are independent of $S(t)$.
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Now, by letting $G(t) = B(t) - D(t)$, then we define the Growth price process $\{S(t); t \geq 0\}$ which is modeled by the geometric progression such that

$$\frac{dX(t)}{dt} = G(t)X(t) \tag{2}$$

where $G(t)$ represents the birth and death diffusion growth rate with external jump process. Consider the Growth rate process $\{G(t); t \geq 0\}$ in which the diffusion coefficient a and the drift coefficient b are both proportional to $G(t)$ at time t. The diffusion process is assumed to be interrupted by external effects occurring at a constant rate c and having magnitudes with distribution $H(\cdot)$. Then $\{G(t); t \geq 0\}$ is a Markov process with State Space $S = [0, \infty)$ and can be regarded as a solution of the stochastic differential equation

$$dG(t) = bG(t)dt + aG(t)dW(t) - G(t^-)dZ(t) \tag{3}$$

Here, $\{W(t)\}$ is a Wiener process with mean zero and variance $\sigma^2 t$. Also, $\{Z(t)\}$ is a compound Poisson process.

$$Z(t) = \sum_{i=1}^{N(t)} Y_i \tag{4}$$

Here $\{N(t)\}$ is a Poisson process with mean rate c, where c is the external jump rate, and $Y_1, Y_2, \ldots$, are independent and identically distributed random variables with distribution function $H(\cdot)$, with mean $\mu = E(Y_1)$ and variance $v^2 = Var(Y_1)$. Note that the moments of $Z(t)$ can be determined from the random sums formulas, and are

$$E[Z(t)] = c\mu t \tag{5}$$

And

$$Var[Z(t)] = (cv^2 + \mu^2)t \tag{6}$$

(cf. Taylor and Karlin (1983), pp. 55, 201)

Now from equation (3) we get,

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$$\frac{dG(t)}{G(t)} = bdt + adW(t) - dZ(t) \qquad (7)$$

Thus, the solution of the stochastic differential equation in (7) is given by

$$G(t) = G(0)\exp\{bt + aW(t) - Z(t)\} \qquad (8)$$

where $G(0)$ is the initial growth rate at time zero.
Rewritting equation (2) we get

$$\frac{dS(t)}{S(t)} = G(t)dt \qquad (9)$$

Taking integral on $[0,t]$ for both sides of equation (9) we get

$$\int_0^t \frac{dS(t)}{S(t)} = \int_0^t G(t)dt$$

Note that

$$\int_0^t \frac{dS(t)}{S(t)} = \ln S(t) - \ln S(0) = \ln \frac{S(t)}{S(0)} \qquad (10)$$

and using Karlin and Taylor (1981) and Al-Eideh and Al-Hussainan (2002) and after some algebraic manipulations, it is easily shown that

$$\int_0^t G(s)ds = \int G(0)\exp\{bs + aW(s) - Z(t)\}ds$$

$$= \frac{2(1-b)}{2a + a^2 - b^2} G(0)\exp\{bt + aW(t) - Z(t)\} \qquad (11)$$

Therefore, the solution of the stochastic differential equation in (9) is given by

$$S(t) = S(0)\exp\left\{\frac{2(1-b)}{2a + a^2 - b^2} G(0)\exp\{bt + aW(t) - Z(t)\}\right\} \qquad (12)$$

where $X(0)$ is the initial population data at time zero.

3. Mean And Variance Approximation of the Growth Price Process S(t) Using the Birth and Death Diffusion Growth Rate Process G(t) With External Jump Process H(.)

In this section, the mean and the variance approximation for the growth price process $S(t)$ using the birth and death diffusion process $G(t)$ with external jump process $H(\cdot)$ defined in equation (12) are derived.
Let $M_1(t) = E[S(t)]$ and $V(t) = V[S(t)]$ be the mean and the variance of $S(t)$ respectively.

Now, using the Manchurian expansion, then $S(t)$ in equation (12) can be approximated by

$$S(t) \approx S(0) + S(0)\left\{\frac{2(1-b)}{2a + a^2 - b^2} G(0)\exp\{bt + aW(t) - Z(t)\}\right\} \qquad (13)$$

or equivalently;

$$S(t) \approx S(0) + S(0)\frac{2(1-b)}{2a + a^2 - b^2}G(t) \qquad (14)$$

where $G(t)$ is defined in equation (8).

Using the results of finding the moment approximation of a birth and death diffusion process with constant rate jump process (cf. Al-Eideh (2001)), it is easily shown that

$$E[G(t)] \approx G(0)\exp\left\{\left(b + \frac{1}{2}a\sigma^2\right)t\right\}\cdot\left\{1 - c\mu t + \frac{1}{2}c(v^2 + \mu^2)t\right\} \qquad (15)$$

and

$$E[G^2(t)] \approx (G(0))^2 \exp\{(2b + 2a^2\sigma^2)t\}\{1 - 2c\mu t + 2c(v^2 + \mu^2)t\} \qquad (16)$$

Therefore, the variance of $G(t)$ is then given by

$$V[G(t)] \approx (G(0))^2 \exp\{(2b + 2a^2\sigma^2)t\}$$

$$\cdot\left\{e^{a^2\sigma^2 t}\left(1 - 2c\mu t + 2c(v^2 + \mu^2)t\right) - \left([1 + c\mu t]^2 + (v^2 + \mu^2)\left[\frac{1}{4}c^2t^2 + ct - c^2\mu t^2\right]\right)\right\} \qquad (17)$$

Now, using the approximated growth price model $S(t)$ in equation (14), we get

$$M_1(t) \approx S(0) + S(0)\frac{2(1-b)}{2a + a^2 - b^2}E[G(t)] \qquad (18)$$

and

$$V(t) \approx (S(0))^2\left(\frac{2(1-b)}{2a + a^2 - b^2}\right)^2 V[G(t)] \qquad (19)$$

where $E[G(t)]$ and $V[G(t)]$ are defined in equations (15) and (17) respectively.

4. Predicted And Simulated Growth Price Model S(t)

In this section, we will obtain the predicted and the simulated sample path of the growth price process $S(t)$ using the birth and death diffusion process $G(t)$ with external jump process $H(\cdot)$ defined in equation (12).

Assuming $M_1(t_n - t_{n-1})$ be the one-step predicted model of $S(t)$, then $M_1(t_n - t_{n-1})$ can be written as

$$M_1(t_n - t_{n-1}) = S(0) + S(0)\frac{2(1-b)G(0)}{2a + a^2 - b^2}\exp\left\{\left(b + \frac{1}{2}a\sigma^2\right)(t_n - t_{n-1})\right\}$$

$$\cdot\left\{1 - c\mu t + \frac{1}{2}c(v^2 + \mu^2)(t_n - t_{n-1})\right\} \qquad (20)$$

For simulation of the growth price process $S(t)$ we used the following discrete approximation.

For integer values $k=1,2,3,\cdots$, and $n=1,2,3,\cdots$, the birth and death growth rate diffusion process $G(t)$ with external jump process $H(\cdot)$ can be simulated by

$$G_n^*\left(\frac{k+1}{n}\right)=G_n^*\left(\frac{k}{n}\right)+\frac{b}{n}G_n^*\left(\frac{k}{n}\right)+\frac{a}{n}G_n^*\left(\frac{k}{n}\right)\cdot Z_{k+1}-G_n^*\left(\frac{k}{n}\right)J\left(\frac{k}{n}\right)\Delta C\left(\frac{k}{n}\right) \tag{21}$$

where $\{Z(k)\}$ is an independent sequence of standard normal random variables and $\Delta C\left(\frac{k}{n}\right);k=1,2,\ldots$ are independent and identically distributed with

$$P\left(\Delta C\left(\frac{k}{n}\right)=1\right)=\frac{c}{n}$$

$$P\left(\Delta C\left(\frac{k}{n}\right)=0\right)=1-\frac{c}{n}$$

and $J\left(\frac{1}{n}\right)$, $J\left(\frac{2}{n}\right)$, ... are independent and identically distributed with distribution $H(\cdot)$.

For each set of positive integers $k,t_1,\ldots,t_k$, the sequence of random vectors $(G_n^*(t_1),\ldots,G_n^*(t_k))'$ converges in distribution to $(G_n(t_1),\ldots,G_n(t_k))'$.

Thus, the simulated Population growth model $S(t)$ is given by

$$S_n^*\left(\frac{k+1}{n}\right)=S_n^*\left(\frac{k}{n}\right)+S_n^*\left(\frac{k}{n}\right)\left\{\frac{2(1-b)}{2a+a^2-b^2}G_n^*\left(\frac{k+1}{n}\right)\right\} \tag{22}$$

where $G_n^*\left(\frac{k+1}{n}\right)$ in defined in equation (21).

Note that for each set of positive integers k, the sequence of random vectors $\left(X_n^*(1),\cdots,X_n^*(k)\right)'$ converges in distribution to $\left(X(1),\cdots,X(k)\right)'$.

5. Numerical Example

Consider, as an example, the following sample paths of the above model $S(t)$ in section (4) that represents the annual price of an item in US dollars when $S(t)=20$, $b=0.02$, $a=2$, $n=20$, and $c=1$ for the following cases:

Case 1

In this case, we consider the sample path to the stochastic growth price model $S(t)$ using the stochastic birth and death diffusion growth rate process $G(t)$ with no external jump process, note in this case the jump rate $c = 0$. Figure 1 and Figure 2 represent this case for $G(t)$ and $S(t)$ respectively.

Figure 1: The Stochastic Growth Rate Process with no External Jump Process

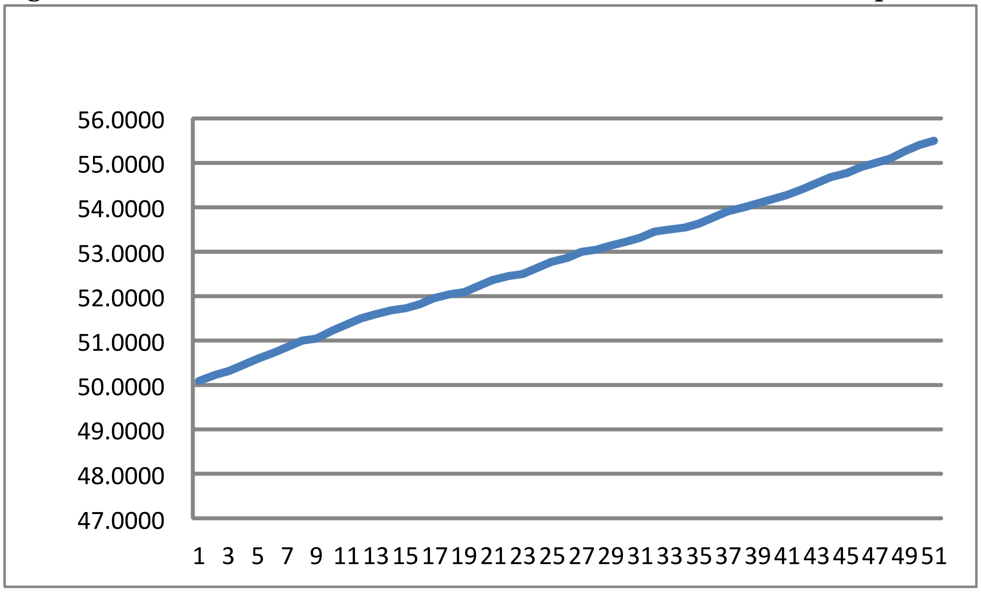
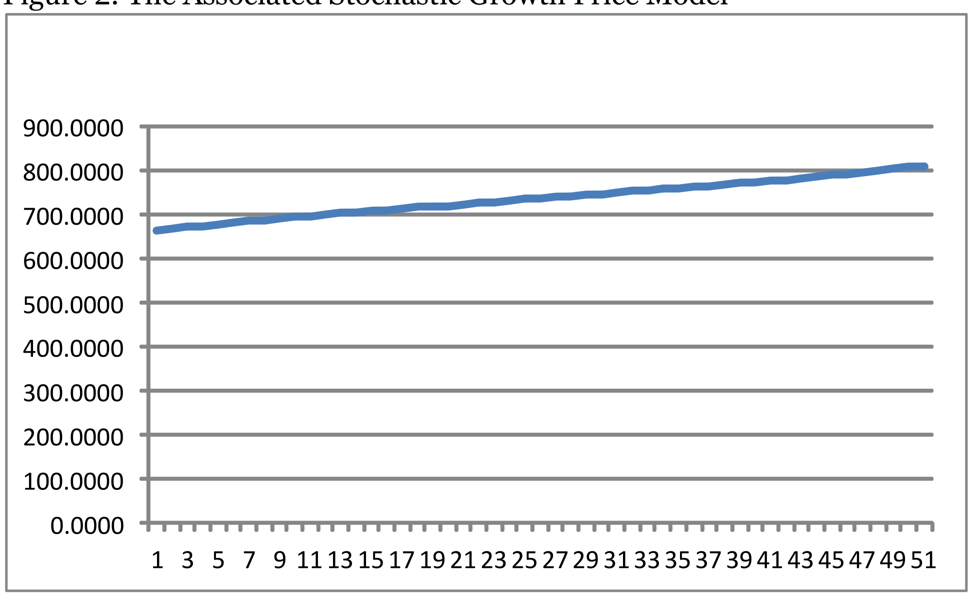


Figure 2: The Associated Stochastic Growth Price Model



Case 2

In this case, we consider the sample path to the stochastic growth price model $S(t)$ using the Stochastic birth and death diffusion growth rate process $G(t)$ with Uniform external jump process, note in this case the jump rate $c = 1$. For simplicity, we take $H(\cdot)$ to be uniform on $[0,1]$. Thus

$$dH(y) = 1, \quad 0 \le y \le 1 \qquad (23)$$

Note that $H(y)$ is independent of y with mean $\mu = \frac{1}{2}$, and variance $v^2 = \frac{1}{12}$. Figure 3 and Figure 4 represent this case for $G(t)$ and $S(t)$ respectively.

Figure 3: The Stochastic Growth Rate Process with Uniform Jump Process

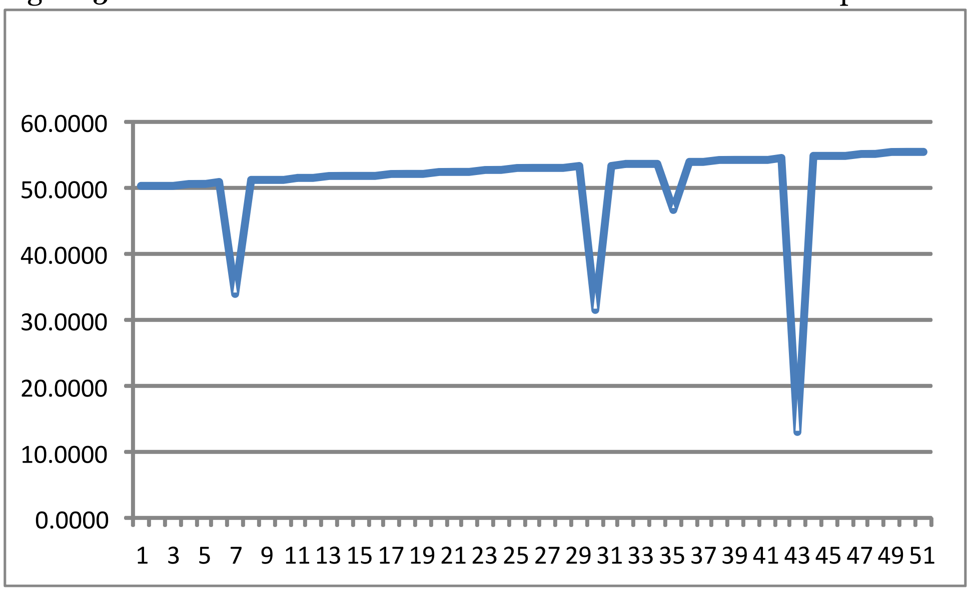
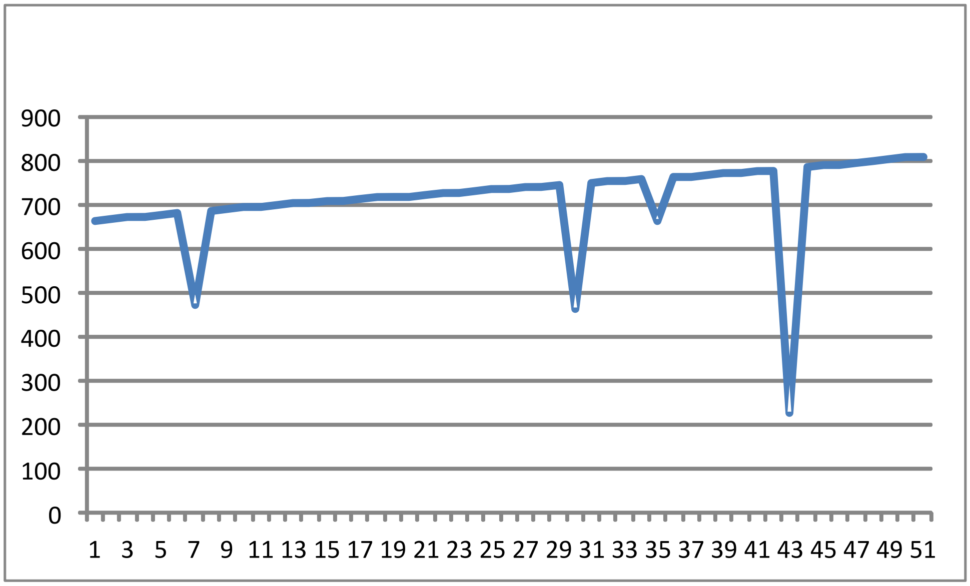


Figure 4: The Associated Stochastic Growth Price Model



Case 3

In this case, we consider the sample path to the stochastic growth price model $S(t)$ using the Stochastic birth and death diffusion growth rate process $G(t)$ with Exponential external jump process, note in this case the jump rate $c = 1$. For simplicity, we take $H(\cdot)$ to be exponential with mean 1. Thus,

$$dH(y) = e^{-y}, \quad y > 0 \tag{23}$$

Note that $H(y)$ depends on y with mean $\mu = 1$, and variance $v^2 = 1$. Figure 5 and Figure 6 represent this case for $G(t)$ and $S(t)$ respectively.

Figure 5: The Stochastic Growth Rate Process With Exponential Jump Process

Figure 6: The Associated Stochastic Growth Price Model

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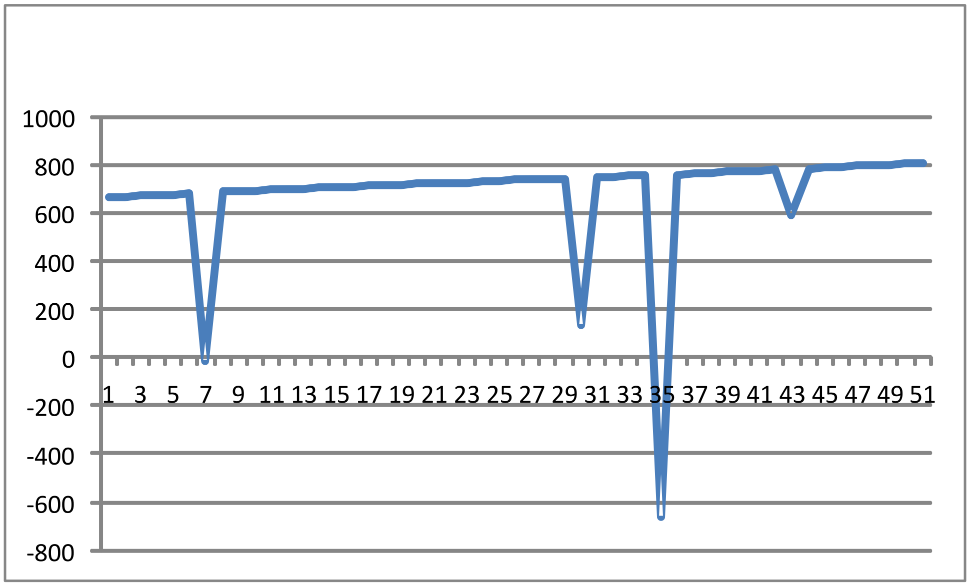

Looking to the above figures, we can see the difference between these figures, also the difference between the uniform jump and the exponential jump is noted in the stochastic growth rate processes and this shows the difference between the jump processes if they are dependent or independent of the growth rates and finally this difference affects the stochastic growth price models. Any way, the figures are reasonable and suggested to be used in the modeling purposes for some prices.

Conclusions

In conclusion, this study provides a methodology for studying the behavior of the prices. More specifically, the study departs from the traditional before and after regression techniques and the time series analysis and developed a stochastic model that explicitly accounts for the variations and volatilities in prices follow a geometric progression using a birth and death diffusion growth rate process subject to randomly occurring external jump processes, especially the uniform on [0, 1] and exponential with mean 1 processes. Ideally, a large class of external jump processes with general jump rate could be tackled in future researches. Also, some inference problems could be done for this model.

In terms of future research, this methodology could be applied not only in prices, but on all aspects of economics and operations research problems.

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